Directed random walk with spatially correlated random transfer rates

Claude Aslangul and Noëlle Pottier
Groupe de Physique des Solides, Tour 23, 2 place Jussieu, 75251 Paris CEDEX 05, France

Petr Chvosta
Institute of Physics of Charles University, Ke Karlovu 5, 121 16 Prague, Czechoslovakia

Daniel Saint-James*
Laboratoire de Physique Statistique, Collège de France, 3 rue d'Ulm, 75231 Paris CEDEX 05, France
(Received 11 May 1992)

We have investigated the random walk of particles in the frame of a conventional master equation for directed random walks. The transfer rates are supposed to be random variables and we incorporate the possibility of correlations. We assume that the chain consists of successive segments of random lengths. Within a given segment, the transfer rates are equal to a single random variable. The transfer rates belonging to two different segments are supposed to be independent and distributed according to the same probability law. We have calculated the time-asymptotic behavior of the mean coordinate of the particle. The resulting character of the motion emerges from the interplay between two basic features: the probability of having a small value of the transfer rate and the probability of having long segments. If the first moment of the segment-length distribution diverges, the asymptotic regime undergoes radical changes as compared to the noncorrelated model.

PACS number(s): 05.40.+j, 05.60.+w, 71.55.−i

I. INTRODUCTION

The study of dynamical features of diffusion or conductivity in random environments has been initiated in early 1980's [1] and intensively pursued during the whole last decade. At present, there exist several review papers on the subject [2–4] and we refer to them for a detailed list of references.

The problem is usually formulated either in a discrete form, i.e., by means of the Pauli master equation (PME) [5] assuming the transfer rates to be random variables, or in a continuous formulation, i.e., using the Fokker-Planck equations [6,7] and assuming the drift function to be a stochastic function in space. The transport properties within the medium are then related to an average over the random parameters in the equation of motion. The averaging process can introduce new features of the resulting dynamics in the time-asymptotic region, the so-called dynamical phases [4].

Most of the above treatments share the common attribute of ignoring correlations in the probabilistic description of the disordered medium. Thus the transfer rates in the PME are usually taken as *independent* random variables and the drift function in the Fokker-Planck equation is supposed to be a *white-noise* process. Nevertheless, as already pointed out in [8], a more realistic description should take into account the possibility of spatial correlations, i.e., the persistence of the ordered elements in a partially randomized medium. Stochastically, this feature can be incorporated by assuming spatial correlations of the parameters which define the local transport properties.

Unfortunately, the problems with correlations are considerably more involved than their noncorrelated counterparts. In the continuous case, one is faced with a stochastic differential equation with a general "colored" noise [9]. In the discrete problem, one cannot use the fairly well-developed theory valid for independent, identically distributed random variables [4,10].

Considering the present status of the problems with correlations, we want to focus on the simplest random-walk problem, namely the *directed* random walk (pure birth process) [6,11–14]. In a first study we used a Fokker-Planck equation as an auxiliary tool for the preparation of a (discrete-"time", continuous-state) Markov chain of correlated random variables with a prescribed stationary distribution [15]. The Markov character of the process enables a simple complete probabilistic description of the system. Yet in this first study the correlations decrease exponentially with the distance. Therefore, in the present work, the emphasis will be put on the consequences of long-range correlations.

The exact solution of the directed "random-random"-walk problem with noncorrelated transfer rates is well understood. The details can be found in Refs. [16,17] (cf. also the papers [18–20] on the problem of general random-random walk) and Ref. [21]. In connection with the existence of the spatial correlations, two basic questions are abetted. First, is the asymptotic regime stable against correlations between the transfer rates? Second, even if the functional form of the asymptotic regime does persist, are the time-scale and/or other parameters of the asymptotic dynamics modified? The authors of Ref. [4] have expressed a conjecture (based on a renormalization-group analysis), namely, that the dynamical phases persist as far as the correlations decrease sufficiently rapidly and are integrable. Another specific model with correlated spatial disorder has been investigated in Ref. [2]. Here the authors analyze a random resistor network with a set of N resistors in series, where the resistance R_j of resistor j changes in a correlated fashion: $R_{j+1}=(1+\epsilon)^{\tau_j}R_j$ with the random variable $\tau_j=\pm 1$. The interplay between the local disorder and the fractal-induced spatial disorder has been examined in Ref. [22]. Otherwise, up to our knowledge, the problem of the spatially correlated disorder has not been treated in the literature.

II. DIRECTED RANDOM-RANDOM WALK

In the problem of directed *random* walk [16,21], the particle is supposed to move along a semi-infinite linear chain only in the direction of increasing coordinate. The motion is described by the Pauli master equation

$$\frac{d}{dt}p_0(t)=-W_0p_0(t) ,$$
$$\frac{d}{dt}p_n(t)=-W_np_n(t)+W_{n-1}p_{n-1}(t) , \quad n\geq 1 , \tag{1}$$

where W_n denotes the transfer rates between the sites of the semi-infinite chain and $p_n(t)$ are the site-occupation probabilities. In the so-called *random-random* walk, the transfer rates are supposed to form a system $\{W_n\}_{n=0}^{\infty}$ of identically distributed, non-negative random variables with the first-order density $\rho(W)$. However, generally, they are not independent, their correlations being described, e.g., by higher-order joint densities.

The system (1) with the initial condition $p_n(0)=\delta_{n0}$ can be easily solved [16]:

$$P_0(z)=\frac{1}{z+W_0} , \quad P_n(z)=\frac{1}{z+W_n}\prod_{k=0}^{n-1}\frac{W_k}{z+W_k} , \quad n\geq 1 , \tag{2}$$

where $P_n(z)$ is the Laplace transform of the occupation probability $p_n(t)$. The transport properties then emerge after carrying out the averaging procedure. As for the properties which are linear in the occupation probabilities, we need the averaged functions

$$G_n(z)=\left\langle\frac{1}{z+W_n}\prod_{k=0}^{n-1}\frac{W_k}{z+W_k}\right\rangle . \tag{3}$$

Here and below, the brackets $\langle\ldots\rangle$ denote the averaging over *all* probabilistic features of the system $\{W_n\}_{n=0}^{\infty}$.

The disorder-averaged motion of the particle is described by the mean coordinate

$$\langle\overline{x(t)}\rangle=\sum_{n=0}^{\infty}n\langle p_n(t)\rangle=\sum_{n=0}^{\infty}ng_n(t) . \tag{4}$$

We are primarily interested in the time-asymptotic behavior of the function $\langle\overline{x(t)}\rangle$. Taking the Laplace transform of $\langle\overline{x(t)}\rangle$, one finds [16]

$$\langle x_1(z)\rangle=\sum_{n=0}^{\infty}nG_n(z)=\frac{1}{z}\sum_{n=0}^{\infty}\left\langle\prod_{k=0}^{n}\frac{W_k}{z+W_k}\right\rangle \equiv\frac{1}{z}\sum_{n=0}^{\infty}\Xi_n(z) . \tag{5}$$

As mentioned above, in the limit case of *independent*, identically distributed random variables the complete description is furnished by the density $\rho(W)$. We shall take it in the form [16]

$$\rho(W)=\begin{cases}\dfrac{\mu}{W_c}\left[\dfrac{W}{W_c}\right]^{\mu-1} & \text{for } W<W_c\\ 0 \text{ for } W>W_c ,\end{cases} \tag{6}$$

where W_c is a sharp cutoff and μ is a positive parameter. The expectation value of W is $W_c\mu/(\mu+1)$ and its variance reads $W_c^2\mu/[(\mu+2)(\mu+1)^2]$. The limit $\mu\rightarrow\infty$ can be regarded as the transition to the ordered (nonrandom) situation, all transfer rates being then equal to W_c. We now proceed to the evaluation of the expression (5). Each term $\Xi_n(z)$ is clearly seen to factorize into the product of $n+1$ functions $S_1(z,\mu)$, where we have adopted the notation

$$S_1(z,\mu)=\left\langle\frac{W}{z+W}\right\rangle=\int_0^{W_c}dW\,\rho(W)\frac{W}{z+w} =\mu\int_0^1 dx\frac{x^\mu}{x+Z} , \tag{7}$$

with $Z=z/W_c$. The expression (5) is a geometrical series, which yields

$$\langle x_1(z)\rangle=\frac{1}{z}\frac{S_1(z,\mu)}{1-S_1(z,\mu)}=\frac{\langle W\rangle}{z^2}\frac{1-S_1(z,\mu+1)}{1-S_1(z,\mu)} . \tag{8}$$

Actually $\langle x_1(z)\rangle$ is the same quantity as in Ref. [16] [part (a), formula (10)], but expressed in a different manner for a further use. The last transcription in (8) already anticipates the general form (21) to be achieved in the model with correlations.

Let us finally recall the behavior for $\langle\overline{x(t)}\rangle$ as deduced from (8) by the Tauberian theorem [12] [see also Eqs. (13)–(17)]

$$\langle\overline{x(t)}\rangle\approx\begin{cases}\dfrac{\sin(\pi\mu)}{\pi\mu\Gamma(\mu+1)}T^\mu & \text{for } \mu<1\\ \dfrac{T}{\ln T} & \text{for } \mu=1\\ \dfrac{\mu-1}{\mu}T & \text{for } \mu>1 ,\end{cases} \tag{9}$$

with $T=W_ct$. A standard regime with constant nonzero asymptotic velocity $v_\infty=\lim_{t\rightarrow\infty}v(t)$ occurs for $\mu>1$: $v_\infty=W_c(\mu-1)/\mu$. On the other hand, if $0<\mu<1$, the coordinate increases slower than t and the asymptotic velocity is zero, $v_\infty=0$. The decisive role is played here by the high weight of the small transfer rates, i.e., by the high probability of quasibroken links [23–25].

Let us now investigate the opposite limit of completely correlated transfer rates. This amounts to assuming that all transfer rates W_n are equal to a single random variable $W_0=W_1=\ldots\equiv W$. Performing the average over an ensemble of ordered chains yields

$$\langle \overline{x(t)}\rangle=\langle W\rangle t=\frac{W_c\mu}{\mu+1}t\ . \tag{10}$$

Note also that the Laplace transform $\langle x_1(z)\rangle$ which corresponds to this result could have been obtained from (5). In the present case the functions $\Xi_n(z)$ are equal to $S_{n+1}(z,\mu)$, where

$$S_k(z,\mu)=\left\langle \left[\frac{W}{z+W}\right]^k\right\rangle=\int_0^{W_c}dW\rho(W)\left[\frac{W}{z+W}\right]^k$$
$$=\mu\int_0^1 dx\frac{x^{k+\mu-1}}{(x+Z)^k}\ . \tag{11}$$

By interchanging the summation over n and the integration over W in the formula (5) we get

$$\langle x_1(z)\rangle=\frac{1}{z}\int_0^{W_c}dW\rho(W)\sum_{n=1}^{\infty}\left[\frac{W}{z+W}\right]^n=\frac{1}{z^2}\frac{W_c\mu}{\mu+1}\ , \tag{12}$$

which is the same result as (10).

The functions $S_k(z,\mu)$ will be extensively used in the sequel and for further use we now give their small-z expansions. The integration required in (11) [26,27] gives for a noninteger $\mu>0$

$$S_k(z,\mu)=1-Z^{\mu}\frac{\pi}{\sin(\pi\mu)}\frac{\Gamma(\mu+k)}{\Gamma(\mu)\Gamma(k)}$$
$$-\mu\sum_{n=1}^{\infty}\frac{(-Z)^n}{n!}\frac{1}{n-\mu}\prod_{i=0}^{n-1}(k+i)\ , \tag{13}$$

where $\Gamma(x)$ is the Euler gamma function. If μ is equal to an integer, $\mu=m=1,2,\ldots$, we find the formulas

$$S_k(z,m)=\frac{1}{(Z+1)^k}-Z\frac{k+m-1}{m-1}S_k(z,m-1)\ ,\quad m=2,3,\ldots, \tag{14}$$

$$S_k(z,1)=1-kZ\ln\left[1+\frac{1}{Z}\right]$$
$$+\sum_{n=2}^{k}\frac{(-1)^n}{(n-1)}\left[\begin{matrix}k\\n\end{matrix}\right]\sum_{i=1}^{n-1}\left[\frac{Z}{Z+1}\right]^i\ . \tag{15}$$

Therefore the small-z behavior of the functions $S_k(z,\mu)$ is

$$S_k(z,\mu)\approx 1-Z^{\mu}\frac{\pi}{\sin(\tau\mu)}\frac{\Gamma(\mu+k)}{\Gamma(\mu)\Gamma(k)}-Z\frac{k\mu}{\mu-1}\ , \tag{16}$$

$$S_k(z,m)\approx 1-kZ\ln\left[\frac{1}{Z}\right]\delta_{m1}-Z\frac{k+m-1}{m-1}(1-\delta_{m1})\ , \tag{17}$$

for noninteger $\mu>0$, and for $\mu=m=1,2,\ldots$, respectively.

III. SIMPLE MODEL FOR CORRELATIONS

We are now ready to introduce the basic assumptions of our model. Considering the sequence of random variables $\{W_n\}_{n=0}^{\infty}$, let us denote as $f_k, k\geq 1$ the probability that the random variable W_k is independent of the random variable W_{k-1}, this case of independence being *the first* to occur in the series $W_0,W_1,\ldots,W_k$. Otherwise stated, the variables preceding this interruption are totally correlated in the sense specified above: $W_0=W_1=\ldots=W_{k-1}$. if any such group of identical variables occurs anywhere in the chain, we shall refer to this faction as to a *segment* (see Fig. 1). Hence, within a given segment, the transfer rates are "ordered," their common value being chosen at random in accordance with the density $\rho(W)$. The length of a segment is equal to an integer k with a probability f_k and the lengths of different segments are statistically independent. If the first interruption in the chain occurs between the variables W_{k-1} and W_k (the probability of this event is just f_k), then the new sequence $\{W_n\}_{n=k}^{\infty}$ is assumed to constitute an exact probabilistic replica of the original sequence $\{W_n\}_{n=0}^{\infty}$. This construction is the spatial equivalent of the renewal process which is well known in probability theory [12,13].

The character of the renewal process critically depends on the large-k form of the probabilities f_k. Introducing the characteristic functions [11,23] for the sequence and for the sequence of tails $\{g_n\}_{n=1}^{\infty}$, $g_n=\sum_{k=n}^{\infty}f_k$, $n=1,2,3,\ldots$

$$f(\theta)=\sum_{n=1}^{\infty}f_n\theta^n\ ,\quad g(\theta)=\sum_{n=1}^{\infty}g_n\theta^n=\theta\frac{1-f(\theta)}{1-\theta}\ , \tag{18}$$

the large-k behavior of the distribution f_k is mirrored by the properties of the function $f(\theta)$ for $\theta\to 1-$. For instance, the mean length of the segments is $d=\lim_{\theta\to 1-}f'(\theta)$, the prime denoting the derivative. A diverging derivative implies an important change of character for the renewal process: the *mean* number of segments which precede a given position, say k, is then asymptotically no longer proportional to k but increases

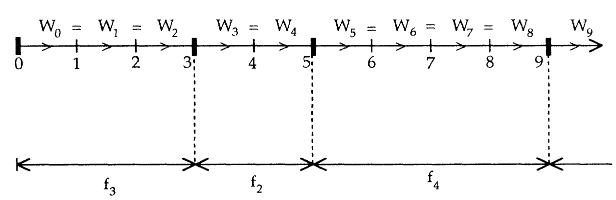

FIG. 1. Arrangement of the transfer rates and of the segments in the directed random-random-walk model. Three segments are shown: the first segment comprises three variables, the second segment two variables, the third segment four variables, etc. Within each segment, the random rates are equal to one common random variable, distributed according to the density (6). For any two different segments, the transfer rates are independent. The lengths of the succeeding segments are independent, identically distributed, integer-valued random variables with a prescribed distribution $\{f_n\}_{n=1}^{\infty}$.

slower than k. This is clearly due to the possibility of encountering longer and longer segments [28].

Let us now return to the expression (5) for the mean coordinate. Taking into account the division into segments as defined above, the terms $\Xi_k(z)$ can be rewritten into the following recursive scheme:

$$\Xi_0(z)=g_1S_1(z,\mu)\ ,$$
$$\Xi_1(z)=f_1S_1(z,\mu)\Xi_0(z)+g_2S_2(z,\mu)\ , \qquad (19)$$
$$\Xi_2(z)=f_1S_1(z,\mu)\Xi_1(z)+f_2S_2(z,\mu)\Xi_0(z)+g_3S_3(z,\mu)\ .$$

We now need to carry out the summation of all these equations. Let us introduce

$$F(z,\mu,\{f_n\})=\sum_{n=1}^{\infty}f_nS_n(z,\mu)=\int_0^{W_c}dW\,\rho(W)f\left[\frac{W}{z+W}\right], \qquad (20)$$

and a similar function $G(z,\mu,\{g_n\})=\sum_{n=1}^{\infty}g_nS_n(z,\mu)$. Solving the algebraic equation for $\langle x_1(z)\rangle$ and using the connection (18) between the characteristic functions $f(\theta)$ and $g(\theta)$, we arrive at the final general expression

$$\langle x_1(z)\rangle=\frac{1}{z}\frac{G(z,\mu,\{f_n\})}{1-F(z,\mu,\{f_n\})}=\frac{\langle W\rangle}{z^2}\frac{1-F(z,\mu+1,\{f_n\})}{1-F(z,\mu,\{f_n\})}\ , \qquad (21)$$

where $\langle W\rangle=W_c\mu/(\mu+1)$. In the subsequent analysis, we will calculate the function $F(z,\mu,\{f_n\})$ for specific distributions $\{f_n\}_{n=1}^{\infty}$. Generally speaking, the small-z expansion of $F(z,\mu,\{f_n\})$ always starts with unity. This follows from the small-z behavior of the functions $S_n(z,\mu)$ as given by Eqs. (16) and (17). As for the higher-order terms, the small-z expansion of $F(z,\mu,\{f_n\})$ will be crucially sensitive to the detailed character of *both* probabilistic aspects of the model, i.e., the small-W form of $\rho(W)$ and the large-n form of f_n.

To close this section, let us now focus on an important ingredient of the model, namely, on the nature of the existing correlations. Given an arbitrary random variable W_n, $n\geq 0$, the variable W_{n+r} with $r\geq 0$ is correlated with W_n if and only if it belongs to the same segment. When this is the case they have actually the same value and their covariance equals to the variance of one of them, that is, $W_c^2\mu/[(\mu+2)(\mu+1)^2]$. However, the event "$W_n$ and W_{n+r} belong to the same segment" occurs only with a certain probability, say $c_n(r)$. On the whole, the covariance (correlation function) within the system $\{W_n\}_{n=0}^{\infty}$ reads

$$\mathrm{cov}\{W_n,W_{n+r}\}=\mathrm{var}\{W_n\}c_n(r)=\frac{W_c^2\mu}{(\mu+2)(\mu+1)^2}c_n(r)\ , \qquad (22)$$

so that for a given μ the correlations are governed solely by the distribution $\{f_n\}_{n=1}^{\infty}$. One should be aware that the construction does not necessarily provide a stationary chain of variables, i.e., $c_n(r)$ generally depends on the index n. Nonetheless, we want to characterize the properties of motion in the asymptotic region in time and in space. In any physically plausible case, the asymptotic form of correlations is given by the behavior of $c_n(r)$ as a function of the range r for high enough n. In this asymptotic region, the n dependence of $c_n(r)$ must be washed out so that the process turns out to be asymptotically stationary. In other words, the physically relevant correlations are related to the r dependence of the limit $c_\infty(r)=\lim_{n\to\infty}c_n(r)$. This quantity denotes the probability that a consecutive sequence of r bonds in the space-asymptotic region is not interrupted by a segment boundary, i.e.,

$$c_\infty(r)=\frac{\sum_{n=1}^{\infty}nf_{n+r}}{\sum_{n=1}^{\infty}nf_n}\ . \qquad (23)$$

The prefactors n in the sums appear because of n possible ways of placing a sequence of r bonds on a segment of length $n+r$ and the denominator guarantees for the proper normalization: $c_\infty(0)=1$. Obviously, care has to be taken if the first moment of the distribution $\{f_n\}_{n=1}^{\infty}$ diverges. This case actually occurs in subsection IV C. Here, the calculation proceeds along a more detailed procedure which makes use of the Tauberian theorem for the ratio of the generating functions $g(r,\theta)/g(\theta)$, where $g(r,\theta)=\sum_{n=1}^{\infty}g_{r+n}\theta^n$ is the "shifted" characteristic function for the sequence of tails.

As for the global characterization of the range of correlations, one usually calculates the *long-range* correlations in the system $\{W_n\}_{n=0}^{\infty}$. In our formulation, they are simply specified by the asymptotic form of the coefficient $c_\infty(r)$ as a function of the index r.

IV. SPECIFIC DISTRIBUTIONS OF LENGTHS OF SEGMENTS

A. Geometric distribution

Suppose that for any neighboring random variables the following assertion holds: with the probability $b\in[0,1]$ the variables W_k and W_{k+1} belong to different segments (i.e., they are independent). This prerequisite implies a geometrical distribution of the segment lengths. Actually, the probability that the first interruption of the run of totally correlated variables will be just between W_{k-1} and W_k is $f_k=ba^{k-1}$, where $a=1-b$. The mean length of the segments is $d=1/b$ and the variance of the length reads $\mathrm{var}=a/b^2$. Introducing an alternative parameter $\alpha>0$, $a=\exp(-\alpha)$, the renewal process is described by the only parameter α and the whole model by the pair μ and α. The characteristic function (18) reads $f(\theta)=b\theta/(1-a\theta)$. The function $F(z,\mu,\{f_n\})$ in Eq. (20) is equal to $S_1(\tilde{z},\mu)$, where $\tilde{z}=z/b$. Eventually, the Laplace transformation of the mean coordinate Eq. (21) assumes the form

$$\langle x_1(z)\rangle=\frac{1}{b^2}\frac{\langle W\rangle}{\tilde{z}^2}\frac{1-S_1(\tilde{z},\mu+1)}{1-S_1(\tilde{z},\mu)}\ . \qquad (24)$$

The only difference with the noncorrelated result (9) consists in the scaling $\langle x_1(z)\rangle \leftrightarrow \langle x_1(\overline{z})\rangle / b^2$, that is, $\langle \overline{x(t)}\rangle \leftrightarrow \langle \overline{x(t)}\rangle / b$. In other words, we do not find any change of the dynamical exponents as compared to the noncorrelated case (9). If $\mu \geq 1$, the asymptotics is exactly the same, if $\mu \in]0,1[$, the present result differs from the noncorrelated one by the prefactor $1/\tau(b)=b^{\mu-1}$. The prefactor diverges for $b \to 0+$, i.e., for diverging mean length of the segments $d=1/b$. This is a prerequisite for the change of the asymptotic regime toward that of the totally correlated case.

Le us specify the existing correlations. The probability that any two random variables W_n and W_{n+r} belong to the same segment is equal to $c_n(r)=\exp(-r\alpha)$ so that $c_\infty(r)=\exp(-r\alpha)$. Due to the independence of $c_n(r)$ on n, we conclude that the process $\{W_n\}_{n=0}^{\infty}$ is exactly stationary from the very beginning (and not just asymptotically stationary).

Incidentally, let us emphasize that the basic presumption of this subsection imposes the *Markovian* character of the relations between the random variables $\{W_n\}_{n=0}^{\infty}$. Indeed, the information that the random variable W_k belongs to a given segment with a specific length *does not* influence the expectation that the variables W_k and W_{k+1} will be uncorrelated. This expectation is always equal to a and it is, e.g., insensitive to the fact that W_k already belongs to a very long segment. This lack of memory is equivalent to the Markovian property, and the system of variables $\{W_n\}_{n=0}^{\infty}$ constitutes a discrete-time continuous-state Markov chain. A more formal construction of the Markov chain [15] rests upon the one-step conditional probability

$$p(W',m+1;W,m)dW' = \mathrm{Prob}\{W_{m+1}\in[W',W'+dW']\,|\,W_m=W\} . \tag{25}$$

For our construction, the conditional probability for the sites n and $n'>n$ reads

$$p(W',n';W,n)=a^{n'-n}\delta(W-W')+(1-a^{n'-n})\rho(W') , \tag{26}$$

where $\rho(W)$ is the first-order density (6). Note that the stationary distribution for the given Markov chain is $\rho(W)$.

B. Poisson distribution

In this subsection, we assume a shifted Poisson distribution for the lengths of segments, that is,

$$f_k=\exp(-\lambda)\frac{\lambda^{k-1}}{(k-1)!} , \quad f(\theta)=\theta\exp[-\lambda(1-\theta)] . \tag{27}$$

$\lambda>0$, $k\geq 1$. The mean length of the segments is $d=1+\lambda$ and the variance of the length equals to var$=\lambda$. If $\lambda\to 0+$ we should recover the case of independent variables. In the opposite limit $\lambda\to\infty$ one has a greater influence of the very long segments. Note that for any λ, all moments of the distribution $\{f_k\}_{k=1}^{\infty}$ exist. The resulting dynamics is described by the two positive parameters μ and λ. Since in this model f_k is no longer a product of step-by-step memoryless individual trials, the chain is *no more Markovian*. The probability of having an interruption on a given site depend on the length of the segment preceding this cut. One could check this property by computing the higher-order joint probability densities for the system $\{W_n\}_{n=0}^{\infty}$.

According to Eq. (21), we first need the function $F(z,\mu,\{f_n\})$, which is related to $S_k(z,\mu)$ by (20). We are interested in the asymptotic behavior of $\langle\overline{x(t)}\rangle$, i.e., in the small-$z$ expansion of the function $F(z,\mu,\{f_n\})$. Since all the moments of the shifted Poisson distribution are finite, we can first use the small-z expansions of $S_k(z,\mu)$ as given by Eqs. (16) and (17), and then perform the summation in (20). Using this device both in the denominator and in the numerator of Eq. (21), we arrive at the following conclusions.

Once more one does not observe any change in the dynamical exponents. If $\mu\geq 1$ we recover exactly the same asymptotic behavior of $\langle\overline{x(t)}\rangle$ with no change of the prefactor. If $\mu\in]0,1[$, there appears a new prefactor $1/\tau(\mu,\lambda)$ in the right-hand side (r.h.s.), in the third formula (9) where

$$\tau(\mu,\lambda)=\frac{\exp(-\lambda)}{\mu(\lambda+1)}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\frac{\Gamma(\mu+k+1)}{\Gamma(\mu)\Gamma(k+1)} = \frac{\exp(-\lambda)}{\lambda+1}\mathcal{M}(1+\mu,1,\lambda) . \tag{28}$$

Here $\mathcal{M}(a,b;x)$ is the Kummer's confluent hypergeometric function [27]. The prefactor goes to unity for $\lambda\to 0+$, recovering thus the noncorrelated situation. In the opposite limit $\lambda\to\infty$, we use the asymptotics of the confluent hypergeometric function [27] and we obtain $1/\tau(\mu,\lambda)\approx\lambda^{1-\mu}\Gamma(\mu)/\mu$. Thus the prefactor diverges for diverging mean length of the segments $d=1+\lambda$ exactly in the same way as for the Markovian case above. Again, this divergence mirrors the transition to the totally correlated regime, that is the change of the asymptotic behavior $t^{\mu}\leftrightarrow t$. On the whole, the only difference with respect to the Markovian case consists in the fact that the prefactor depends on *both* parameters μ and λ. Apart from this, all the important features of the asymptotic behavior of the mean coordinate $\langle\overline{x(t)}\rangle$ are identical.

As for the correlations within the system $\{W_n\}_{n=0}^{\infty}$, one gets

$$c_\infty(r)=\frac{\exp(-\lambda)}{1+\lambda}\frac{\lambda^r}{r!}\mathcal{M}(2,r+1,\lambda) \overset{r\to\infty}{\approx} \frac{1}{\sqrt{2\pi}}\frac{\exp(-\lambda)}{1+\lambda}\exp\left[-r\ln\left[\frac{r}{\lambda}\right]\right] . \tag{29}$$

This formula gives the desired physically relevant correlations in the space-asymptotic region and the *long-range* correlations in this region. Due to the logarithmic correction in the exponent, the correlations decrease more rapidly than in the Markovian case. This conclusion is clearly a direct consequence of the quicker damping of the probability for having long segments. Eventually, let us check the correlations in the limit of large mean length of the segments $d=1+\lambda$, that is $\lambda\to\infty$. Coming back to the first expression in Eq. (29) one obtains $c_\infty(r)\approx\lambda/(1+\lambda)$. Therefore, as expected on physical grounds, the asymptotic correlations do not depend on the distance (index r) and the correlation coefficient goes to unity.

C. Beta distribution

As stated above, the overall character of the process $\{W_n\}_{n=0}^{\infty}$ is radically modified when the first moment of the distribution $\{f_k\}_{k=1}^{\infty}$ diverges. In order to investigate the implications of this feature within our model, let us analyze the distribution

$$f_k=\sigma\mathcal{B}(k,\sigma+1)=\sigma\frac{\Gamma(k)\Gamma(\sigma+1)}{\Gamma(k+\sigma+1)}\approx\frac{\sigma^2\Gamma(\sigma)}{k^{\sigma+1}}\left[1+\frac{\sigma(\sigma+1)}{k^2}+O(k^{-2})\right], \tag{30}$$

where $\mathcal{B}(x,y)$ is the beta function [27] and the asymptotic formula reveals the probability of having very long ordered segments. The first moment is finite only if $\sigma>1$. The choice of this particular distribution is made to facilitate the calculations. The corresponding characteristic function (18) reads

$$f(\theta)=\sigma\int_0^1 dy\, y^{\sigma-1}\frac{y\theta}{1-\theta+y\theta}=\frac{\sigma}{\sigma+1}\frac{\theta}{1-\theta}\mathcal{F}\left[1,1+\sigma,2+\sigma;-\frac{\theta}{1-\theta}\right], \tag{31}$$

where $\mathcal{F}(a,b,c;z)$ is the Gauss hypergeometric function [27]. On the whole, the present detailed form of our model is fully described by the couple of parameters μ and σ.

The calculation proceeds as before except that one has to make the summation before the small-z expansion. Presently, we insert the above integral representation of $f(\theta)$ into the last expression in Eq. (20) for $F(z,\mu,\sigma)$ and obtain:

$$1-F(z,\mu,\sigma)=\mu\sigma\int_0^1 dx\, x^{\mu-1}\int_0^1 dy\, y^{\sigma-1}\frac{Z}{Z+xy}=\mu\sigma\sum_{n=0}^{\infty}\frac{(-Y)^n}{(n+\mu)(n+\sigma)}, \tag{32}$$

with $Z=z/W_c$ and $Y=1/Z$. For $|Y|<1$, the series is convergent. We need an equivalent asymptotic series in $1/Y$ (i.e., the small Z expansion). The most direct method for deriving such asymptotic representation is by way of Mellin transforms [28], that is, by considering the integral representation

$$1-F(z\mu,\sigma)=\mu\sigma\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}d\omega\, Y^{-\omega}\times\frac{1}{(\omega-\mu)(\omega-\sigma)}\frac{\pi}{\sin(\omega\pi)}. \tag{33}$$

Here we impose the condition $0<\gamma<\min(1,\mu,\sigma)$, and we integrate along the contour which envelops the left (negative) half plane anticlockwise, thus recovering the expansion (33). Alternatively one can close the contour clockwise in the positive half plane and one obtains another expansion

$$1-F(z,\mu,\sigma)=-\mu\sigma\sum_n \mathrm{Res}_{\omega=\omega_n}\left[Z^{\omega}\frac{\pi}{\sin(\omega\pi)}\frac{1}{(\omega-\sigma)(\omega-\mu)}\right]. \tag{34}$$

The summation runs over all poles of the bracketed function and the residues can be calculated by usual complex analysis (notice the possibility of double and triple poles). For the illustrative purposes, we consider only the most general case when the parameters μ and σ do not coincide and when they are both different from a positive integer. A short calculation based on (34) then establishes the formula

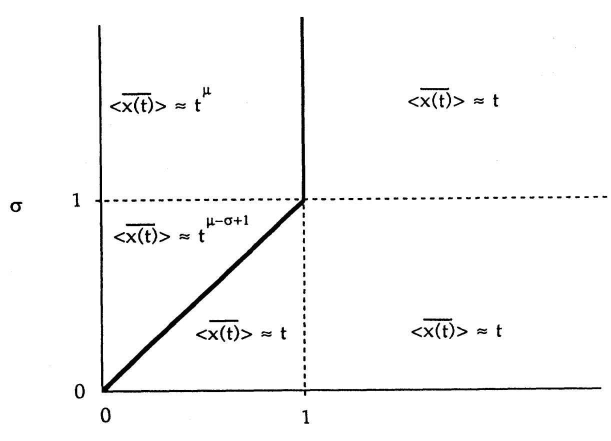

FIG. 2. Asymptotic behavior of the mean coordinate $\langle\overline{x(t)}\rangle$ in the (μ,σ) plane as derived in subsection IV C. The parameter μ describes the small-W behavior of the probability density for the transfer rates (6). In the noncorrelated model, the anomalous phase $\langle\overline{x(t)}\rangle\approx T^{\mu}$ exists for $\mu<1$. The parameter σ specifies the β distribution (30) for the segment lengths. For $\sigma<1$, the correlations between the transfer rates are not damped to zero.

$$1-F(z,\mu,\sigma) = -\mu\sigma\left\{\sum_{n=1}^{\infty}\frac{(-Z)^n}{(n-\mu)(n-\sigma)} + \frac{\pi}{(\mu-\sigma)}\left[\frac{Z^\mu}{\sin(\mu\pi)}-\frac{Z^\sigma}{\sin(\sigma\pi)}\right]\right\}. \quad (35)$$

The series converges for $|Z|<1$, i.e., for $|Y|>1$ and the small-z behavior thereof only depends on the mutual relation of the numbers μ, σ, and 1. In general, the presence of the double or triple poles only brings about logarithmic corrections to the multiform terms Z^μ and Z^σ.

Having at hand the small-z expansion of the numerator and the denominator in the formula (21), we again proceed directly to the Tauberian asymptotic analysis of the coordinate $\langle\overline{x(t)}\rangle$ itself. The complete summary of the results is the following ($T+W_c t$ is again the reduced-time variable):

$$\langle\overline{x(t)}\rangle \approx \begin{cases} \dfrac{\mu-1}{\mu}T & \text{for } \sigma\geq 1\ , \quad \mu>1 \\ \dfrac{T}{\ln T} & \text{for } \sigma\geq 1\ , \quad \mu=1 \\ \dfrac{(\sigma-\mu)\sin(\pi\mu)}{(\sigma-1)\pi\mu\Gamma(\mu+1)}T^\mu & \text{for } \sigma>1\ , \quad \mu<1 \\ \dfrac{(1-\mu)\sin(\pi\mu)}{\pi\mu\Gamma(\mu+1)}T^\mu\ln T & \text{for } \sigma=1\ , \quad \mu<1 \\ \dfrac{\mu-\sigma}{1+\mu-\sigma}T & \text{for } \sigma<1\ , \quad \mu\geq\sigma \\ \dfrac{T}{\ln T} & \text{for } \sigma<1\ , \quad \mu=\sigma \\ \dfrac{(\sigma-\mu)\sin(\pi\mu)}{(1+\mu-\sigma)\sin(\pi\sigma)\Gamma(2+\mu-\sigma)}T^{\mu+1-\sigma} & \text{for } \sigma<1\ , \quad \mu<\sigma\ . \end{cases}$$

Figure 2 presents the asymptotic behavior of $\langle\overline{x(t)}\rangle$ in the plane (μ,σ). Comparing these results with those of the noncorrelated model Eq. (9) we observe that when the mean length of segments does not diverge ($\sigma>1$), the only modification is a prefactor. However, when the mean length diverges the dynamical exponents are modified and even a normal behavior $\langle\overline{x(t)}\rangle\approx T$ may be recovered.

Let us again investigate the nature of the underlying correlations as a careful treatment of the ratio which has been mentioned in connection with Eq. (23) yields the result

$$c_\infty(r)\approx\begin{cases}\sigma\Gamma(\sigma)\dfrac{1}{r^{\sigma-1}} & \text{for } \sigma>1\ , \quad r\to\infty \\ 1 \quad \text{for } \sigma\leq 1\ , \quad r\to\infty\ . \end{cases} \quad (36)$$

For $\sigma>1$, the correlations tend to zero when $r\to\infty$ and in this case no modification of the asymptotics of $\langle\overline{x(t)}\rangle$ (except for the prefactor) is observed, even if the correlation are not integrable (i.e., when $1<\sigma<2$). On the other hand when the correlations are not damped to zero ($\sigma\leq 1$) the normal behavior $\langle\overline{x(t)}\rangle\approx T$ is recovered only when the disorder is small enough (i.e., $\mu>\sigma$). For $\mu<\sigma$ one still observes an anomalous dynamical phase, but "accelerated," $T^\mu\to T^{\mu+(1-\sigma)}$.

V. CONCLUSIONS

The present work has been aimed to analyze the dynamical consequences of spatial correlations in a disordered medium. For the purpose of reviewing the essence of our construction, it will be expedient to contrast it against the classical model developed in [25]. These authors have investigated the vibration frequency spectrum of disordered lattices, assuming the presence of ordered "islands" of light atoms separated by randomly distributed "walls" of rigid atoms with infinite mass.

First, in our construction, the segments are not ordered in the above sense, their attribute being the equality of transfer rates as random variables. Therefore, every segment is described by one random variable with a given prescribed density. Second, the boundary between the segments is not marked by a zero value of the transfer rate but instead by the property of statistical independence. Third, the boundaries between the segments are distributed randomly according to a prescribed integer-value distribution; the latter is not necessarily restricted to the simple Bernoulli-trials distribution.

An alternative description of the interplay between the local and the spatial (or "constitutional") disorder can be traced in Ref. [22]. Here, the authors investigate continuous-time random walk CTRW processes on fractals. The local disorder is determined by the specific form of the waiting-time distribution within the usual form of the CTRW formalism [29,30]. This representation can be directly connected with our PME method, which operators with the random transfer rates [31]. However, contrary to our segmentlike stochastical construction, the spatial disorder in [22] rests on the topological complexity of the substrate and it is depicted by the fractal and spectral (fracton) dimensions of the underlying fractal structure. One observes a different role of the parameters which describe the two types of the disorder in the final dynamical predictions.

The most important results of the present work are as follows (all of them should be understood as valid in the space- and time-asymptotic region). When the correlations decrease with the distance, whatever the detailed type of damping, the dynamical exponents as observed in the noncorrelated model are retained. The correlation only shows itself by the presence of an additional prefactor in the noncorrelated asymptotics. Moreover, the prefactor is only present in the region of the anomalous dynamical phase of the noncorrelated model, i.e., for $\mu < 1$. The structure of the prefactor reflects the details of the distribution of the segment lengths and the details of the assumed probability density for the transfer rates.

The dynamical exponents can be modified only if the correlations do not decrease to zero with distance. First, this feature is present if there exists a nonzero probability of having an infinitely long segment. Second, we have detected the change of the dynamical exponents (as compared to the noncorrelated model) in the case where the reman length of the segments diverges. Here, the probability of having very long ordered segments decreases slowly with the length of the segments. A direct consequence of this effect is the absence of the damping of correlations with the range and, second, the modification of the dynamical exponents as detailed in subsection IV C. The anomalous behavior is accelerated and even, if the disorder is small enough, the normal behavior is recovered. Finally, notice that in our construction, the dynamical exponents *are not* changed if the correlations decrease algebraically as $r^{-(\sigma-1)}$, $\sigma > 1$. This observation applies also in the case of nonintegrated correlations, i.e., for $\sigma < 2$. This conclusion for the present *directed* random walk is in contrast with a prediction based on a renormalization-group analysis of a zero-bias model in [4].

ACKNOWLEDGMENTS

One of the authors (P.Ch.) would like to thank the University Paris VII for financial support and to express his gratitude for the warm hospitality extended to him at the "Groupe de Physique des Solides." The Groupe de Physique des Solides is "associé au Centre National de la Recherche Scientifique URA No. 17 et aux Universités Paris VII et Paris VI."

---

*Also at Université Paris VII, Paris, France.

[1] S. Alexander, J. Bernasconi, W. R. Schneider, and R. Orbach, Rev. Mod. Phys. **53**, 175 (1981).

[2] S. Havlin and D. Ben-Avraham, Adv. Phys. **36**, 695 (1987).

[3] J. W. Haus and K. W. Kehr, Phys. Rep. **150**, 263 (1987).

[4] J. P. Bouchaud and A. Georges, Phys. Rep. **195**, 127 (1990).

[5] N. G. Van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1981).

[6] C. W. Gardiner, *Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences,* 2nd ed. (Springer, Berlin, 1985).

[7] H. Risken, *The Fokker-Planck Equation: Methods of Solution and Applications* (Springer, New York, 1984).

[8] B. Derrida, J. Stat. Phys. **433**, 31 (1983).

[9] N. G. Van Kampen, J. Stat. Phys. **54**, 1289 (1989).

[10] B. V. Gnedenko and A. N. Kolmogorov, *Limit Distributions of Sums of Independent Random Variables* (Addison Wesley, Reading, MA, 1954).

[11] W. Feller, *An Introduction to Probability Theory and Its Applications,* 3rd ed. (Wiley, New York, 1968), Vol. I.

[12] W. Feller, *An Introduction to Probability Theory and Its Applications* (Ref. [11]), Vol. II.

[13] S. Karlin and H. M. Taylor, *A First Course in Stochastic Processes* (Academic, New York, 1975).

[14] A. Papoulis, *Probability, Random Variables, and Stochastic Processes,* 2nd ed. (McGraw-Hill, New York, 1984).

[15] C. Aslangul, P. Chvosta, N. Pottier, and D. Saint-James, Europhys. Lett. (to be published).

[16] C. Aslangul, M. Barthélémy N. Pottier, and D. Saint-James, J. Stat Phys. **59**, 11 (1990).

[17] C. Aslangul, M. Barthélémy, N. Pottier, and D. Saint-James, J. Stat. Phys. **61**, 403 (1990).

[18] C. Aslangul, N. Pottier, and D. Saint-James, J. Phys. (Paris) **50**, 899 (1989).

[19] C. Aslangul, N. Pottier, and D. Saint-James, Physica A **164**, 52 (1990).

[20] C. Aslangul, M. Barthélémy, N. Potteier, and D. Saint-James, Physica A **171**, 47 (1991).

[21] E. Buffet and P. Hannigan, J. Stat. Phys. **65**, 645 (1991).

[22] A. Blumen, J. Klafter, B. S. White, and G. Zumofen, Phys. Rev. Lett. **53**, 1301 (1984).

[23] E. W. Montroll and G. H. Weiss, J. Math. Phys. **6**, 167 (1965).

[24] B. D. Hughes, Physica A **134**, 443 (1986).

[25] C. Domb, A. A. Maradudin, E. W. Montroll, and G. H. Weiss, Phys. Rev. **115**, 24 (1959).

[26] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic, New York, 1981).

[27] *Handbook of Mathematical Functions,* edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1970).

[28] R. B. Dingle, *Asymptotic Expansions: Their Derivation and Interpretation* (Academic, London, 1973).

[29] J. Klafter and R. Silbey, J. Chem. Phys. **72(2)**, 843 (1980).

[30] M. Sahimi, B. D. Hughes, L. E. Scriven, and H. Ted Davis, J. Chem. Phys. **78(11)**, 6849 (1983).

[31] V. M. Kenkre, E. W. Montroll, and M. F. Shlesinger, J. Stat. Phys. **9**, 45 (1973).